REPLACEMENT OF AUXILIARY EXPRESSIONS

1. *Auxiliary and nonauxiliary expressions.* Certain expressions are sometimes regarded as in some sense *auxiliary*. For example, some physicists regard words like "electron" not as names of real objects but rather as useful symbolic devices. Some psychologists think of words like "ego," "purpose," etc., which do not refer to directly observable entities, as "intervening variables" which help to correlate patterns of behavior. Nominalists usually regard expressions involving reference to classes as mere manners of speaking. To take a final example, some phenomenalists believe that sentences about the external world serve merely to order our sense experiences. In all these cases, which expressions are regarded as auxiliary (and also in exactly what sense and to what degree) depends on the point of view adopted. For example, "measuring rod" may be auxiliary to phenomenalists and non-auxiliary to physicists, and "electron" may be auxiliary to some physicists but not to others.

2. *Replacement programs.* Rightly or wrongly, persons with empiricist leanings sometimes have misgivings concerning expressions which they regard as auxiliary. This leads them sometimes to propose that such expressions be replaced, or at least shown to be replaceable, by expressions which they regard as nonauxiliary and thus somehow "safer." For example, some phenomenalists reject sentences about the external world as "meaningless" unless they can be translated into "equivalent" sentences about sense perceptions. They therefore urge a program of showing that such translations are always possible. Also, for example, Bridgman and other physicists propose that words like "electron" be "operationally defined." Proposals of this kind we shall call *replacement programs*. Some of these programs demand the actual replacement of the expressions regarded as auxiliary, while others only demand a proof of the possibility of such replacement. We shall not distinguish between them. By a *solution* of a replace-

ment program we shall therefore likewise understand either actual replacement or a proof of the possibility of replacement, depending on the demands of the particular program.

3. *Purposes of this paper.* The main purpose of this paper is to describe in a less technical and condensed manner than has been done elsewhere[1] a method of solving certain replacement programs. This method seems applicable whenever the *formulation* of the program possesses two features (to be discussed below in sections 4 and 6) without which the program seems to have little chance of success. It should be added at once, however, that the method is artificial and that the solutions it yields are philosophically quite unsatisfactory.

Another purpose of this paper is to emphasize certain aspects of how replacement programs should be formulated. The discrepancy between how they should be and how they are formulated will then be apparent. This discrepancy, which is perhaps inevitable, is another reason why the present method is probably at most of theoretical interest.

4. *Co-operative Programs. Effective dichotomy.* In general, a replacement program is proposed without being solved immediately. Other persons are asked to participate in working toward a solution. The program is intended to be a *co-operative enterprise.*

For this purpose it should be clearly stated, among other things, which expressions are regarded as auxiliary and which as nonauxiliary. Indeed the class of expressions regarded as auxiliary should be defined *effectively*, i.e., *in a manner which allows any one, given any expression, to determine in a finite number of steps whether or not the expression belongs to the class.* Otherwise we might encounter an expression for which we know of no way of finding out whether or not it is regarded as auxiliary and hence whether or not we should try to replace it. Also, the complementary class of expressions regarded as nonauxiliary should be defined effectively.[2] Otherwise an expression might be proposed as a replacement for which we

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[1] "On Axiomatizability within a System," *Journal of Symbolic Logic*, XVIII (1953), 30-32. I am grateful to Professor C. G. Hempel for his encouragement and advice to give a less technical presentation.

[2] Actually, if one class is effectively defined, its complementary class is also effectively defined.

know of no way of finding out whether or not it is regarded as nonauxiliary and hence whether or not it serves our purpose. If auxiliary and nonauxiliary expressions are thus effectively distinguished, we shall say that an *effective dichotomy* has been given.

It should be noted that in practice a clear-cut distinction between auxiliary and nonauxiliary expressions is often hard to make or else artificial. For example, in the beginning it may have been more natural to regard the term "virus" as auxiliary. Nowadays it may be more natural to regard it as nonauxiliary. Yet the shift from one category to the other seems gradual, not abrupt. Even the first microscopic observation of viruses would not seem to warrant a sudden shift.

5. *Objective Formulation.* In formulating a program that is intended to be co-operative, it should also be specified exactly what is meant by a correct solution. These specifications should leave no room for guesswork or individual interpretation. They should be drawn up in such a way that if and only if the specifications are satisfied, this can be demonstrated objectively, i.e., in a manner carrying universal conviction. Programs provided with such specifications we shall call *objectively formulated.*

Without objective formulation, disputes may arise as to whether or not a program has actually been solved. Programs of an abstract nature, including replacement programs, lend themselves particularly to such disputes. Also, since the very purpose of a replacement program is clarification, the program itself should be formulated with a maximum of clarity and hence, if possible, objectively. Finally, without a clear conception of what is meant by a correct solution, the chances of finding one are much poorer.

Objective formulation, while highly desirable, is perhaps not absolutely essential. Even without it, some one may conceivably stumble on a solution which happens to be universally accepted. Also, a program may be stimulating even if it has no solutions.

6. *Systematization.* So much for objective formulation of any program. Let us now see more specifically what is required for objectively formulating a replacement program.

The purpose of a replacement program is to clarify a subject matter, not to change it. One of the requirements therefore which a solution must satisfy before we can regard it as correct is that it leaves unchanged what is regarded to be the essential content of the subject matter.[3] An objective formulation of a replacement program should make it clear exactly when this requirement is satisfied. It should therefore also make it clear exactly what is regarded as the essential content of the subject matter.

In general, the best way of specifying what one regards to be the essential content of the subject matter is to specify the class T of all the sentences which one regards to be true assertions of the subject matter, separating afterwards those sentences one regards as essential from the others. In order to leave no room for guess-work, one must specify exactly what one regards as a *proof* that a sentence belongs to T. Moreover, one must specify these proofs in a manner which allows us, given any argument, to decide in a finite number of steps whether or not the argument constitutes such a proof; otherwise, a proof might fail in its function, which is to settle matters once and for all. More briefly, one must define the class of such proofs effectively.

If one has specified T in this manner then one has in effect constructed a formal system (for details see Section 7 below) whose theorems constitute T. Our discussion can therefore be summarized as follows: In order to formulate a replacement program objectively, one must in general construct a formal system whose theorems constitute what one regards as the true assertions of the subject matter.

In practice, it should be obvious, few subject matters can be systematized in this manner. Even in a fairly rigorous subject such as physics the speed with which the subject changes and the coexistence of competing theories, among other factors, seem to make systematization practically impossible.

7. *Conditions for formal systems.* Consider any formal system S whose theorems are intended to express the content of a certain subject matter other than logic. Ordinarily, we distinguish in this case between the *extralogical axioms* of S and the *underlying logic*, L, of S. To L belong all the *rules of inference* of S, i.e., all rules whereby from certain sentences of S we can infer others. To L

[3] Certain modifications of the content, such as occur in Carnap's "explications," are perhaps permissible. They can perhaps be construed as affecting only the nonessential content.

also may or may not belong certain axioms. These axioms, if any, are the *logical axioms* of *S*.

We now list certain conditions which any formal system *S* must satisfy before it is ordinarily regarded as an adequate formalization of a given subject matter.

(i) The class of applications of a rule of inference of *S* is effectively defined.
(ii) The class of logical axioms of *S* is effectively defined.
(iii) The class of extralogical axioms of *S* is effectively defined.
(iv) The class of sentences of *S* is effectively defined.
(v) Every axiom of *S* is either logical or extralogical.
(vi) No axiom of *S* is both logical and extralogical.
(vii) Every theorem of *S* is a sentence of *S*.

Conditions (ii), (iii), and (v) together imply that the class of axioms of *S* is effectively defined. Together with (i) this implies that the class of proofs in *S*, i.e., proofs which are carried out in *S*, is effectively defined.

The underlying logic *L* of *S*, consisting of the rules of inference and the logical axioms of *S*, may itself be regarded as a formal system. The theorems of *L* are those expressions for which there exists a proof in *L*. Conditions (i) and (ii) imply that the class of proofs in *L* is effectively defined.

8. *Conditions for replacing entire proofs.* Replacement programs demand that expressions regarded as auxiliary be replaced (or at least shown to be replaceable) everywhere. For a given subject matter, this requires their replacement not only in single statements but also in entire demonstrations or proofs.[4] Thus no replacement program has been solved unless, for the given subject matter, the original class of proofs is replaced by a new class of proofs.

In order to state more precisely certain conditions which the new class of proofs should satisfy, we shall restrict ourselves from now on to replacement programs whose formulation possesses the following two features, suggested by the discussion up to now:

(a) A formal system *S* satisfying (i) to (vii) is given whose theorems constitute what are regarded to be the true assertions of the given subject matter.

[4] This aspect of replacement programs has in general been overlooked.

(b) An effective dichotomy is given which distinguishes among the expressions of S those which are regarded as auxiliary from those which are not.

Given (a), replacement of the original class of proofs for a given subject matter by a new class of proofs amounts to replacement of the original system S by a new system S^*. Since auxiliary expressions must be replaced everywhere, one condition which S^* must satisfy is as follows:

(1) No sentence of S^* contains expressions regarded as auxiliary.

Before considering a second condition on S^*, we shall restrict ourselves further to replacement programs of the following kind:

(c) If a sentence contains an expression regarded as auxiliary, then the sentence itself is regarded as auxiliary.

(d) If a sentence is regarded as auxiliary, then it is also regarded as inessential.[5]

Restrictions (c) and (d) together imply that a sentence is regarded as essential only if it contains no expressions regarded as auxiliary. Restriction (c) seems to be satisfied by most replacement programs. For example, if the word "electron" is regarded as auxiliary in the sense of serving merely as a device for organizing sentences about observations, then any sentence containing the word "electron" is usually regarded as a similar device and hence as auxiliary in the same sense. To motivate restriction (d), one would have to make more precise in which sense the word "essential" is used. Nevertheless, in most senses of the two words a sentence which is regarded as auxiliary is also regarded as inessential. We shall leave the word "essential" intentionally vague and shall make no further restrictions concerning the relationship (often very complex) between those sentences which are regarded as inessential and those expressions which are regarded as auxiliary. For example, we shall admit the possibility that the sentence "Either it will rain tomorrow or it won't" is regarded as inessential even if it does not contain expressions regarded as auxiliary.

[5] The distinction between essential and inessential seems to apply not only to assertions regarded as true but also to other sentences. For example, Fermat's "Last Theorem," which has neither been proved nor disproved, may be regarded by mathematicians as essential in the sense that its truth would be important.

A second condition which S^* must satisfy is that it and S agree in those portions of the content they express which are regarded as essential. Given that the content expressed by a system is expressed by its theorems and given restrictions (c) and (d), this condition is satisfied whenever

(2) S and S^* agree in those of their theorems which contain no expressions regarded as auxiliary.

For convenience, (2) may be broken up into two conditions:

(2a) *Consistency condition on S^* (relative to S):* If A is a theorem of S^* and contains no expressions regarded as auxiliary[6], then A is also a theorem of S.

(2b) *Completeness condition on S^* (relative to S):* If A is a theorem of S and contains no expressions regarded as auxiliary, then A is also a theorem of S^*.

A third condition which S^* must satisfy before the replacement program can be regarded as solved may be roughly stated as follows: The underlying logic L^* of S^* should differ from the underlying logic L of S as little as possible. In more detail, L^* should in no respect be stronger than L, and weaker only to the extent required by (1). To state this condition more precisely, we must consider the rules of inference of L, since the strength of a logic depends on its rules as well as on its theorems (see footnote 1 to the Appendix). We shall say that a rule of inference R is *valid in L* if and only if addition of R to L would not strengthen L. More precisely, by R being valid in L we shall mean that either R is a rule of inference of L, or else any sentence which can be deduced from other sentences and from axioms of L by applications of rules of inference of L, together with applications of R, can also be deduced from these sentences and from axioms of L by applications only of rules of inference of L, without applications of R. Our condition on the underlying logic L^* of S^* can now be stated as follows:

(3a) *Consistency condition on L^* (relative to L):* Any rule of inference of L^* is valid in L. Moreover, any theorem of L^* is a theorem of L.

(3b) *Completeness condition on L (relative to L):* Any rule of

[6] The second clause is automatically satisfied when S^* satisfies (vii) and (1).

inference of L, restricted in its application and yield to sentences without expressions regarded as auxiliary, is valid in L^*. Moreover, any theorem of L which contains no expressions regarded as auxiliary is a theorem of L^*.

The consistency condition (3a) prevents us, among other things, from compensating for the lack in S^* of expressions regarded as auxiliary by adopting additional, and perhaps highly artificial, rules of inference. That we could thus compensate has been shown recently.[7]

9. *A method of solving certain replacement programs.* We shall now describe a method of solving certain replacement programs. The method applies only to programs which satisfy the above restrictions (a) to (d) as well as three minor restrictions (e) to (g) described below. The method yields a solution only in this sense: It allows us to replace S by a system S^* which satisfies conditions (i) to (vii) for a formal system and conditions (1) to (3) for replacing entire proofs.

The three additional restrictions on S are as follows:

(e) The rule "If A, then A & A & . . . & A," where "&" is a symbol for conjunction, is valid in the underlying logic L of S. (It follows that L contains a symbol for conjunction.)

(f) The converse rule "If A & A & . . . & A, then A" is also valid in L. We shall call it the rule of *simplification.*

(g) The given dichotomy is such that if no expression in A is regarded as auxiliary, then no expression in A & A & . . . & A is regarded as auxiliary.

Restriction (g) is natural. Restrictions (e) and (f) are satisfied by most formal systems ordinarily encountered. Moreover, other restrictions which sometimes hold when (e) or (f) fails can be substituted for (e) and (f).[8]

Suppose now that (a) to (g) hold. To prepare for our solution of the replacement program, we assign to each sequence of symbols of the given system S one among the numbers 1, 2, 3, . . . , making sure that no number is assigned to more than one sequence

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[7] Hans Hermes, "Zum Begriff der Axiomatisierbarkeit," *Math. Nachrichten*, IV (1950-1951), 343-347.

[8] Details can be found in the article mentioned in footnote 1.

and that, given any number, we either can reconstruct the sequence to which it has been assigned or else eventually find out that it has not been assigned to any sequence. Details of how this is done can be found in many books on mathematical logic. A number thus assigned is a *Gödel number* of the sequence. In particular, any proof in S is a sequence of symbols of S and hence has a Gödel number. Hence also, in particular, any proof in L has a Gödel number.

Consider now any conjunction A & A & . . . & A. We can determine in a finite number of steps whether or not

(α) The number of occurrences of A in A & A & . . . & A is a Gödel number of a proof in L,

(β) This proof is a proof of A, and

(γ) A contains no expressions regarded as auxiliary.

For suppose that A & A & . . . & A is given. Then we first count the number n of occurrences of A in it. We next determine whether or not n is a Gödel number of a sequence of symbols of S. If it is, we can then determine by (i) and (ii) whether or not the sequence is a proof in L, i.e., whether or not (α) holds. If (α) holds we can then test for (β). Finally, since according to (b) an effective dichotomy has been given, we can test for (γ).

Similarly, given A & A & . . . & A we can test whether or not

(α') The number of occurrences of A in A & A & . . . & A is a Gödel number of a proof in S which is not a proof in L,

and whether or not (β) and (γ) hold. For, by (i), (ii), (iii), and (v) we can test whether or not a number is the Gödel number of proof in S, and by (i) and (ii) we can test whether or not n fails to be the Gödel number of a proof in L. Hence we can test for (α'). We can then test for (β) and (γ) as above.

We now form a system S^* and its underlying logic L^* as follows: As the axioms of L^*, i.e., as logical axioms of S^*, we choose all those and only those conjunctions A & A & . . . & A for which (α), (β), and (γ) hold. As rules of inference of L^* we choose the rule of simplification and the rules of inference of L, restricting their application and their yield to sentences which contain no expressions regarded as auxiliary. As extralogical axioms of S^*, we choose all those and only those conjunctions A & A & . . . & A for which (α'), (β), and (γ) hold. Finally, as sentences of S^* we choose those sentences of S which contain no expressions regarded as auxiliary.[9]

10. *Three examples.* We shall now consider three applications of the method just described. In all three cases, the given replacement programs are idealizations of actual programs.

For a given science, the distinction between expressions regarded as auxiliary and expressions regarded as nonauxiliary often amounts to a distinction between predicates expressing what are regarded as *observable* properties and predicates regarded as *theoretical*. For example, in physics the predicate "weighing more than 3 lbs." may be regarded as expressing an observable property while the predicate "consisting of 2 protons and 2 neutrons" may be regarded as serving a theoretical function.

We shall therefore now consider a replacement program whose formulation satisfies (a) to (g), such that the effective dichotomy provided according to (b) is one between predicates regarded as expressing observable properties and predicates regarded as theoretical. Then the system S^* obtained by our method yields all those and only those theorems of S in which all predicates are regarded as expressing observable properties. The "theoretical superstructure" of S has been discarded in S^* without changing what is regarded to be the observable content.[10] Predicates regarded as theoretical are therefore in a certain sense dispensable.

In this example, the underlying logic L^* of S^* differs from the underlying logic L of S only in that its theorems and its rules of inference are restricted to sentences without predicates regarded as theoretical. Thus L^* is essentially the same as L except that from its extralogical vocabulary those portions which are not needed for S^* have been removed.

As a second example, we shall consider an idealized nominalistic replacement program. Nominalists believe that only individuals exist, although they seem to differ in what they regard as individuals. Linguistically, this gives rise to a distinction between

[9] The proof that S^* satisfies conditions (i) to (vii) and (1) to (3), which is somewhat technical, is given in the appendix.

[10] This content is expressed by some, but not necessarily all, of those theorems of S in which all predicates are regarded as expressing observable properties.

variables ranging over what are regarded as individuals and variables ranging over other objects. It also gives rise to a distinction between predicates applying to what are regarded as individuals and predicates applying to other objects.

We shall therefore now consider a replacement program whose formulation satisfies (a) to (g), such that the effective dichotomy provided according to (b) makes the distinctions just indicated. Then by our method we can construct a system S^* whose theorems are all those and only those theorems of S which are *nominalistically acceptable*, i.e., in which all variables range over and all predicates apply to objects regarded as individuals. S^* extracts, so to speak, the nominalistic content of S.

In this example, the relation of the underlying logic L^* of S^* to the underlying logic L of S is more complex. The theorems of L^* are those of L which are nominalistically acceptable. Some logicians, however, may regard other theorems of L as important logical truths and hence may regard L^* as deficient relative to L.

As regards rules of inference, L^* may be deficient relative to L even in the eyes of a nominalist. Condition (3b) guarantees only that every nominalistically acceptable rule of inference which is an explicit rule of L is valid in L^*. It does not guarantee the stronger but also desirable property that every nominalistically acceptable rule of inference which is valid in L is also valid in L^*. Indeed, it is doubtful that this property can always be achieved. For consider a deduction of a nominalistically acceptable sentence from other such sentences by means of axioms of L and rules of inference of L. Some axioms or rules used in this deduction may be nominalistically unacceptable and hence may be without an analogue in L^*. Hence it may be impossible to deduce the given sentence from the other sentences by means of axioms of L^* and rules of inference of L^*.

The desirable property just discussed seems nevertheless to be achieved whenever the formulation of L includes a formulation of the classical first order predicate calculus. In that case, L^* also includes a formulation of this calculus. Then from the completeness of this calculus it would seem to follow that, whenever L is one of the ordinary logics, then any nominalistically acceptable rule of inference which is valid in L is also valid in L^*.

As a third example, we shall briefly consider what may be regarded as an oversimplified version of a positivistic replacement program. Some positivists reject as "meaningless" many sentences which contain both universal and existential quantifiers, since the universal quantifier usually precludes verification and the existential quantifier usually precludes falsification.

We shall therefore now consider a replacement program whose formulation satisfies (a) to (g), such that the effective dichotomy provided according to (b) is one between sentences which contain both existential and universal quantifiers and other sentences. Then our method yields a system S^* whose theorems are all those and only those theorems of S which do not contain both universal and existential quantifiers.

In this example, the underlying logic L^* of S^* is much weaker than any ordinary logic. For example, in contrast to ordinary logics, L^* contains no theorems of the form $(\mathrm{E}x)(y)\mathrm{R}xy \rightarrow (y)(\mathrm{E}x)\mathrm{R}xy$.

11. *Two undesirable properties of* S^*. The problem of replacing S by S^* was essentially a problem of axiomatization, namely of finding axioms for those theorems of S which contain no expressions regarded as auxiliary. Ordinarily, the main purpose of axiomatizing a set of theorems is to express their content in a form which is psychologically or mathematically more perspicuous. Now in the system S^* there corresponds to every theorem A an axiom A & A & . . . & A. Hence the set of axioms of S^* is not more perspicuous than the set of theorems. The axioms fail to simplify or to provide genuine insight. This failure seems to be the principal objection to the present method. Although the method allows us to avoid the use of expressions regarded as auxiliary, nothing seems to be gained thereby.

The great complexity of the axioms of S^* is basically due to the mechanical and artificial way in which they are produced. This manner of production also results in several other undesirable features. We shall briefly discuss one of them.

Suppose that A & A & . . . & A is an extralogical axiom of S^*. This implies according to (α') that there exists a proof of A in S which is not a proof in the underlying logic L. This does not rule out, however, the existence of some other proof of A which is a proof in L. In that case, of course, L^* contains an axiom of the form A & A & . . . & A. Hence it is possible that S^* contains an extralogical axiom and a logical axiom which are both conjunctions of the same A. Then S^* would violate the spirit, although not the letter, of condition (vi), since the intention of (vi) is to establish a clear-cut line between the underlying logic and the extralogical assumptions. Unfortunately, this violation of the intention of (vi) seems unavoidable in some cases, no matter how we may try to modify the construction of S^*.

12. *Answer to a criticism.* One criticism of the method of replacing S by S^* may be stated as follows: The use of expressions regarded as auxiliary is not really avoided, because in order to construct axioms of S^* we must construct proofs in S and therefore use expressions regarded as auxiliary.

This criticism can be answered as follows: The method described in Section 9 of testing whether or not a given conjunction A & A & . . . & A is an axiom of S^* can be modified so as to bypass the actual construction of proofs in S. Instead, we can test by a purely numerical method. Details of this method do not matter here. We must specify, among other things, which numbers are Gödel numbers of axioms, which numbers are Gödel numbers of expressions regarded as auxiliary, etc. What matters is that in order to specify these Gödel numbers we do not have to write out the axioms, the expressions regarded as auxiliary, etc.; we only have to use *names* for them. Now names for expressions regarded as auxiliary seem no more objectionable than names for other expressions. In either case, we are naming the same kind of object.

13. *Replacement of individual expressions.* We shall now discuss another possible criticism. It may be stated as follows: The present method yields replacements only for entire systems. It fails to yield replacements for individual expressions, i.e., individual phrases or sentences. Hence it constitutes at best only a partial solution of a replacement program.

Although this criticism seems justified, we shall now argue that replacement of individual expressions does not always seem possible and that replacement of a system as a whole seems sometimes the best we can do. We shall first state two conditions which seem to be satisfied whenever we regard an expression N as a correct replacement for an expression M. In the first place, we seem to accept the following condition, which may be thought of as a rule of inference yielding new truths from sentences already regarded as true:

(4) Any sentence which results from a sentence regarded as true by replacement of M by N is itself regarded as true.

In the second place we seem to think that replacement of M by N only clarifies but does not change what we regard as the essential content of the given subject matter. Hence the following condition also seems to be satisfied:

(5) Using (4) as a rule of inference does not enlarge what is regarded as the essential content of the given subject matter.

To make these two conditions more precise, we shall now again suppose that those sentences concerning the given subject matter which are regarded as true form the theorems of a formal system S. Moreover, we shall suppose that those sentences of S which are regarded as essential have been distinguished from the others. Then (4) and (5) together imply the following condition:

(6) Any sentence which is regarded as essential and which results from a theorem of S by replacement of M by N is itself a theorem of S.

For let P be any such sentence. Since any theorem of S is regarded as true, P results from a sentence regarded as true by replacement of M by N. Hence, according to (4), P is regarded as true. On the other hand, if P were not a theorem of S then it would not be regarded as true without using (4) as a rule of inference. Hence if P were not a theorem of S, then the use of (4) as a rule of inference would enlarge the class of sentences regarded as true. Since, moreover, P is regarded as an essential sentence, this would enlarge what is regarded as the essential content of the given subject matter. Hence if P were not a theorem of S, condition (5) would be violated.

We now see why replacement of individual expressions appears to be sometimes impossible. Condition (6) is so strong that it seems to fall little short of the requirement that the identity $M = N$ (or, if M is a sentence, the equivalence $M \equiv N$) is a theorem of S. It is therefore quite unlikely that, no matter what M regarded as auxiliary is given, we can always find an $\mathcal{N}$ regarded as non-auxiliary satisfying (6).[11]

14. *Conclusion.* In conclusion, it must be admitted that the picture presented here of the motivation behind replacement programs has been one-sided. Empiricists do not only want to show that expressions regarded as auxiliary can be replaced. Their main aim is rather to clarify somehow the meaning of such expressions. Apart from its other short-comings, the method discussed in this paper fails to provide any such clarification.

On the other hand, our description of the difficulties of a replacement program also applies to any more ambitious program which includes a replacement program as part. In particular, for any such program there remain the difficulties of providing an objective formulation and an effective dichotomy. Also there remain the difficulties pointed out in Section 13 of replacing individual expressions. On the whole, therefore, the outlook for a program which includes a replacement program is dim.

The natural way out of these difficulties is to formulate empiricist programs in a manner which no longer includes replacement programs. This seems justified because it now seems doubtful that the notion of what is empirically significant coincides with any of the notions of replaceability. On the one hand, Section 13 makes it plausible that some individual expressions are not replaceable and yet are empirically significant. On the other hand, according to Section 9, any system is replaceable, no matter what its content.

Looking at the matter a little more closely, it appears that empirical significance attaches to an entire framework of assertions or beliefs, and to individual expressions or concepts only indirectly, by means of that framework. The empirical significance of such a framework seems to depend on how reliably (not on how simply) related it is to experience. Thus empirical significance seems to be a matter of degree.

Without a replacement program, the aims of an empiricist program are less spectacular but perhaps more valuable. These

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[11] For discussion of a related problem see A. Tarski, "Einige methodologische Untersuchungen über die Definierbarkeit der Begriffe," *Erkenntnis,* V (1935), 80-100.

aims are to stand guard against frameworks of beliefs whose relation to experience is unreliable, and also patiently to improve and cement the relation to experience of those beliefs which do seem worth while.

15. *Self-sufficiency of formal languages.* The result of Section 9 can be stated in a way which may be of interest for philosophy of language. From now on the underlying logic of a formal system will be called a *formal language* to stress the fact that not only its axioms and rules of inference but also its primitive symbols and sentences must be specified. The same formal language is often shared by many different formal systems and is, in a sense, more basic than the systems. For example, a system of Euclidean and a system of non-Euclidean geometry may share the same basic language of geometry, e.g., a system of applied first order predicate calculus.

Some classes T^* of sentences of a formal language L^* may be specified by adding (effectively defined) axioms to L^* and thus forming a system S^* such that T^* is the class of theorems of S^*. In this case, T^* may be called *axiomatizable within* L^*. In other cases, it may be more convenient to resort to a language L which is richer than L^* in containing additional sentences, axioms, or rules of inference. T^* may then sometimes be specified by adding (effectively defined) axioms to L and thus forming a system S such that T^* is the set of those theorems of S which are sentences of L^*. In that case, T^* may be called *axiomatizable by going beyond* L^*. By reasoning similar to that of Section 9[12] it can be shown that any language L^* in which both the rule of simplification and its converse are valid, and hence any formal language ordinarily encountered, is *self-sufficient* in the following sense: Any class of sentences of L^* which is axiomatizable by going beyond L^* is also axiomatizable within L^*.

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[12] In defining the extralogical axioms we must replace (α) and (γ) by the following respectively:

(α″) The number of occurrences of A in A & A & . . . & A is the Gödel number of a proof in S which is not a proof in L^*.

(γ″) A is a sentence of L^*.

APPENDIX

From Section 9 it is plain that S^* satisfies condition (1). We shall now show that S^* also satisfies conditions (i) to (vii) and conditions (2) and (3). In Section 9 it was shown that we can determine in a finite number of steps whether or not an expression is an axiom of L^*, i.e., a logical axiom of S^*. Hence S^* satisfies (ii). Similarly, the class of extralogical axioms of S^* is effectively defined, i.e., S^* satisfies (iii). S^* also satisfies (iv), i.e., the class of sentences of S^* is effectively defined. For consider any given expression. Since S satisfies (iv), we can determine whether or not the expression is a sentence of S. If it is, we can then determine by (b) whether or not it contains no expressions regarded as auxiliary, i.e., whether or not it is a sentence of S^*. Furthermore, S^* satisfies (i). For consider any application of a rule of inference. We can determine whether or not it is an application of rule of inference of S, since S satisfies (i). We can also determine whether or not it is an application of the rule of simplification. Finally, as has just been shown, we can determine whether or not each of the sentences involved contains no expressions regarded as auxiliary. We can therefore determine whether or not the given application is an application of a rule of inference of S^*.

From the way S^* was formed it is evident that every axiom of S^* is either logical or extralogical and hence that S^* satisfies (v). S^* also satisfies (vi), since no proof in S which is not a proof in L has the same Gödel number as a proof in L. Therefore any logical axiom of S^* differs from any extralogical axiom of S^* in the number of occurrences of terms in the conjunction.

We now want to show that any theorem B of S^* is a sentence of S^*, i.e., that S^* satisfies (vii). Suppose first that B is a logical axiom of S^*. Then by (α), (β), and (γ), B is a conjunction A & A & . . . & A, where A is a theorem of S and contains no expressions regarded as auxiliary. By (g), A & A & . . . & A also contains no expressions regarded as auxiliary. Moreover, by (e), A & A & . . . & A is a theorem of S and therefore, since S satisfies (vii), a sentence of S. Similarly, if B is an extralogical axiom of S^*, then by (α'), (β), and (γ), B is a sentence of S and contains no expressions regarded as auxiliary. Therefore every axiom of S^* is a sentence of S containing no expressions regarded as auxiliary. Moreover, application of any of the rules of inference of L^* yields only sentences with the same property. Therefore any theorem of S^* is a sentence of S containing no expressions regarded as auxiliary, and thus a sentence of S^*.

Consider any theorem A of S^*. We now want to show that A is also a theorem of S and hence that S^* satisfies the consistency condition (2a). Suppose first that A is an axiom of S^*. Then, as we have just shown in the proof of (vii), A is a theorem of S. Suppose now that A is not an axiom of S^*. Then A is obtained from axioms of S^*, and hence from theorems of S, by applications of rules of inference of L^*. Each of these rules is either also a rule of inference of L or else is the rule of simplification and therefore, by (f), valid in L. Therefore A is a theorem of S.

The proof that each theorem of L^* is also a theorem of L is similar. Moreover, any rule of inference of L^* is valid in L, since either it is a rule of inference of L or else it is the rule of simplification, which is valid in L. It follows that L^* satisfies the consistency condition (3a).

Consider any theorem A of S which contains no expressions regarded as auxiliary. We now want to show that A is also a theorem of S^* and hence that S^* satisfies the completeness condition (2b). Since A is a theorem of S it has a proof in S. This proof has a Gödel number, say n. Then the conjunction A & A & . . . & A with exactly n occurrences of A satisfies either (α), (β), and (γ) or (α'), (β), and (γ), and hence it is either a logical or an extralogical axiom of S^*. Then A can be inferred form this axiom by the rule of simplification and therefore is a theorem of S^*.[1]

In a similar manner we can prove that each theorem of L which contains no expressions regarded as auxiliary is also a theorem of L^*. Moreover, any rule of inference of L, when restricted in its application and yield to sentences without expressions regarded as auxiliary, is a rule of inference of L^* and therefore valid in L^*. It follows that L^* satisfies the completeness condition (3b). This concludes the proof that S^* satisfies conditions (i) to (vii) and (1) to (3).

W. C.

[1] Note that only the rule of simplification is needed for this purpose. All other rules of inference of L^* are therefore redundant in L^* and in S^* in the sense that they yield no additional theorems. Nevertheless, the strength of a logic is ordinarily measured not only by its theorems but also by what the consequences in it are of sentences which are not theorems, i.e., what would be theorems in it if sentences other than theorems were assumed to be true. In this sense then these other rules of inference ordinarily strengthen L^* as follows: From certain sentences of L^* which are not theorems certain others are deducible with these rules which are not deducible with the rule of simplification alone. For example, if *modus ponens* is among the rules of L^*, then from p and $p \rightarrow q$ we can always deduce q, whereas we cannot always deduce q by simplification.